

Light-front vacuum and instantons in two dimensions

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Abstract

We review several aspects of Yang-Mills theory (YMT) in two dimensions, related to its perturbative and topological properties. Consistency between light-front and equal-time formulations is thoroughly discussed.

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I. INTRODUCTION

Non-abelian quantum gauge theories are still far from being satisfactorily understood. Though some non-perturbative features are thought to be transparent, a consistent framework in the continuum is lacking.

Therefore one often resorts to the simplified context of two-dimensional theories where exact solutions can sometimes be available. In two dimensions the theory looks seemingly trivial when quantized in the light-cone gauge (LCG) $A_- \equiv \frac{A_0 - A_1}{\sqrt{2}} = 0$. As a matter of fact, in the absence of dynamical fermions, no physical local degrees of freedom appear in the Lagrangian.

Still topological degrees of freedom occur if the theory is put on a (partially or totally) compact manifold, whereas the simpler behavior on the plane enforced by the LCG condition entails a severe worsening in its infrared structure. These features are related aspects of the same basic issue: even in two dimensions ($D = 2$) the theory contains some non-trivial dynamics. We can say that, in LCG, dynamics gets hidden in the very singular nature of correlators at large distances (IR singularities).

In order to fully appreciate this point and the controversial aspects related to it, let us briefly review the 't Hooft's model for QCD_2 at large N , N being the number of colours [1]. In LCG no self-interaction occurs for the gauge fields; in the large- N limit planar diagrams dominate, without quark loops. The $q\bar{q}$ interaction is mediated by the exchange

$$\mathcal{D}(x) = -\frac{i}{2}|x^-|\delta(x^+), \quad (1)$$

which looks instantaneous if x^+ is considered as a *time* variable. Eq.(1) is the Fourier transform of the quantity

$$\tilde{\mathcal{D}}(k) = \frac{1}{k_-^2}, \quad (2)$$

the singularity at $k_- = 0$ being interpreted as a Cauchy principal value. Such an expression in turn can be derived by quantizing the theory on the light front (at equal x^+), A_+ behaving as a constraint [2].

The full set of ladder diagrams can easily be summed, leading to a beautiful pattern of $q\bar{q}$ -bound states with squared masses lying on rising Regge trajectories. This was the first evidence, to our knowledge, of a *stringy* nature of QCD in its confining regime, reconciling dual models with a partonic field theory.

Still, if the theory within the same gauge choice is canonically quantized *at equal times*, a different expression is obtained for the exchange in eq.(1)

$$\mathcal{D}_c(x) = \frac{1}{2\pi} \frac{x^-}{-x^+ + i\epsilon x^-}, \quad (3)$$

and its Fourier transform

$$\tilde{\mathcal{D}}_c(k) = \frac{1}{(k_- + i\epsilon k_+)^2}, \quad (4)$$

can now be interpreted as a *causal* Feynman propagator [2].

This expression, first proposed by Wu [3], is nothing but the restriction at $D = 2$ of the prescription for the LCG vector propagator in four dimensions suggested by Mandelstam and Leibbrandt [4] (ML), and derived in ref. [5] by equal-time canonical quantization of the theory.

In dimensions higher than two, where “physical” degrees of freedom are switched on (transverse “gluons”), this causal prescription is mandatory in order to get correct analyticity properties, which in turn are the basis of any consistent renormalization program [6].

When eq.(4) is used in summing the very same set of planar diagrams considered by ’t Hooft, no rising Regge trajectories are found in the spectrum of the $q\bar{q}$ -system. The bound-state integral equation looks difficult to be solved; early approximate treatments [7] as well as a more detailed recent study [8] indicate the presence of a massless solution, with a fairly obscure interpretation, at least in this context. Confinement seems lost.

Then, how can it be that the causal way to treat the infrared (IR) singularities, which is mandatory in higher dimensions, leads to a disastrous result when adopted at $D = 2$? In order to get an answer we address ourselves to the $q\bar{q}$ -potential.

II. THE WILSON LOOP

A very convenient gauge invariant way of looking at the $q\bar{q}$ -potential is to consider a rectangular Wilson loop, centered at the origin, with sides parallel to a spatial direction and to the time direction, of length $2L$ and $2T$ respectively

$$\mathcal{W} = \frac{1}{N} \langle 0 | \text{Tr} \left[\mathcal{T} \mathcal{P} \exp \left(ig \oint_{\gamma} dx^{\mu} A_{\mu}(x) \right) \right] | 0 \rangle, \quad (5)$$

the symbols \mathcal{T} and \mathcal{P} denoting temporal ordering of operators and colour ordering.

It is well known that the Wilson loop we have hitherto introduced can be thought to describe the interaction of a couple of static $q\bar{q}$ at the distance $2L$ from each other. We can turn to the Euclidean formulation replacing T with iT . If we denote by $\mathcal{E}_0(L)$ the ground state energy of the system, we get for large T

$$\mathcal{W} = \exp[(4m - 2\mathcal{E}_0)T] \int_{\mathcal{E}_0}^{\infty} d\mathcal{E} \rho(L, \mathcal{E}) \exp[-2T(\mathcal{E} - \mathcal{E}_0)]. \quad (6)$$

Unitarity requires the spectral density $\rho(L, \mathcal{E})$ to be a non-negative measure. Then \mathcal{W} is positive and the coefficient of the exponential factor $\exp[(4m - 2\mathcal{E}_0)T]$ is a non-increasing function of T .

We can define the $q\bar{q}$ -potential as

$$\mathcal{V}(L) = \mathcal{E}_0(L) - 2m.$$

If the theory confines, $\mathcal{V}(L)$ is an increasing function of the distance L ; if at large distances the increase is linear in L , namely $\mathcal{V}(L) \simeq 2\sigma L$, we obtain an area-law behaviour for the leading exponent with a string tension σ .

For $D > 2$ perturbation theory is unreliable in computing the true spectrum of the $q\bar{q}$ -system. However, when combined with unitarity, it puts an intriguing constraint on the $q\bar{q}$ -potential. To realize this point, let us consider the formal expansion

$$\mathcal{V}(L) = g^2 \mathcal{V}_1(L) + g^4 \mathcal{V}_2(L) + \dots, \quad (7)$$

g being the QCD coupling constant.

When inserted in the expression $\exp[-2\mathcal{V}(L)T]$, it gives

$$\exp[-2T\mathcal{V}] = 1 - 2T[g^2\mathcal{V}_1 + g^4\mathcal{V}_2 + \dots] + 2T^2[g^4\mathcal{V}_1^2 + \dots] + \dots \quad (8)$$

At $\mathcal{O}(g^4)$, the coefficient of the leading term at large T should be half the square of the term at $\mathcal{O}(g^2)$. This constraint has often been used as a check of (perturbative) gauge invariance.

Therefore, if we denote by $C_{F(A)}$ the quadratic Casimir expression for the fundamental (adjoint) representation of $SU(N)$ and remember that \mathcal{V}_1 is proportional to C_F , at $\mathcal{O}(g^4)$ the term with the coefficient $C_F C_A$ should be subleading in the large- T limit with respect to the Abelian-like term, which is proportional to C_F^2 .

Such a calculation at $\mathcal{O}(g^4)$ for the loop \mathcal{W} has been performed using Feynman gauge in [9], with the number of space-time dimensions larger than two ($D > 2$). The result depends on the area $\mathcal{A} = 4LT$ and on the dimensionless ratio $\beta = \frac{L}{T}$. The $\mathcal{O}(g^2)$ -term is obviously proportional to C_F ; at $\mathcal{O}(g^4)$ we find that the non-Abelian term is indeed subleading

$$T^2 \mathcal{V}^{na} \propto C_F C_A \mathcal{A}^2 T^{4-2D}. \quad (9)$$

Therefore agreement with exponentiation holds and the validity of previous perturbative tests of gauge invariance in higher dimensions is vindicated.

The limit of our result when $D \rightarrow 2$ is *finite* and depends only on \mathcal{A} , as expected on the basis of the invariance of the theory in two dimensions under area-preserving diffeomorphisms. However the non-Abelian term is no longer subleading in the limit $T \rightarrow \infty$, as it is clear from eq.(9); we get instead [9]

$$2T^2 \mathcal{V}^{na} = C_F C_A \frac{\mathcal{A}^2}{16\pi^2} \left(1 + \frac{\pi^2}{3}\right). \quad (10)$$

We conclude that the limits $T \rightarrow \infty$ and $D \rightarrow 2$ *do not commute*.

This result is confirmed by a calculation of \mathcal{W} performed in LCG with the ML prescription for the vector propagator [10]. At odds with Feynman gauge where the vector propagator is not a tempered distribution at $D = 2$, in LCG the calculation can also be performed directly in two space-time dimensions. The result one obtains does *not* coincide with eq.(10). One gets instead

$$2T^2 \mathcal{V}^{na} = C_F C_A \frac{\mathcal{A}^2}{48}. \quad (11)$$

The extra term in eq.(10) originates from the self-energy correction to the vector propagator. In spite of the fact that the triple vector vertex vanishes in two dimensions in LCG, the self-energy correction does not. We stress that this “anomaly-like” contribution is not a pathology of LCG, it is needed to comply with Feynman gauge.

Perturbation theory is *discontinuous* at $D = 2$. We conclude that the perturbative result, no matter what gauge one adopts, conflicts with unitarity in two dimensions.

Taking advantage of the invariance under area-preserving diffeomorphisms in dimensions $D = 2$, Staudacher and Krauth [11] were able to generalize our $\mathcal{O}(g^4)$ result (eq.(11)) by fully resumming the perturbative series. In the Euclidean formulation, which is allowed as the causal propagator can be Wick-rotated, and with a particular choice of the contour (a circumference), they get

$$\mathcal{W}(\mathcal{A}) = \frac{1}{N} \exp \left[-\frac{g^2 \mathcal{A}}{4} \right] L_{N-1}^{(1)} \left(\frac{g^2 \mathcal{A}}{2} \right), \quad (12)$$

the function $L_{N-1}^{(1)}$ being a Laguerre polynomial.

This result can be further generalized to a loop winding n -times around the contour

$$\mathcal{W} = \frac{1}{N} \exp \left[-\frac{g^2 \mathcal{A} n^2}{4} \right] L_{N-1}^{(1)} \left(\frac{g^2 \mathcal{A} n^2}{2} \right). \quad (13)$$

From eq.(12) one immediately realizes that, for even values of N , the result is no longer positive in the large- T limit. Moreover in the 't Hooft's limit $N \rightarrow \infty$ with $g^2 N = 2\hat{g}^2$ fixed, the string tension vanishes and eq.(12) becomes

$$\mathcal{W} \rightarrow \frac{1}{\sqrt{\hat{g}^2 \mathcal{A}}} J_1(2\sqrt{\hat{g}^2 \mathcal{A}}), \quad (14)$$

J_1 being the usual Bessel function. Confinement is lost.

This explains the failure of the Wu's approach in getting a bound state spectrum lying on rising Regge trajectories in the large- N limit.

However in LCG the theory can also be quantized on the *light-front* (at equal x^+); with such a choice, in pure YMT and just in two dimensions, no dynamical degrees of freedom

occur as the non vanishing component of the vector field does not propagate, but rather gives rise to an instantaneous (in x^+) Coulomb-like interaction (see eq.(1)).

Only planar diagrams contribute to the Wilson loop \mathcal{W} for any value of N , thanks to the “instantaneous” nature of such an exchange; the perturbative series can be easily resummed, leading to the result (for imaginary time)

$$\mathcal{W}(\mathcal{A}) = \exp \left[- \frac{g^2 N \mathcal{A}}{4} \right], \quad (15)$$

to be compared with eq.(12).

Not only is this result in complete agreement with the exponentiation required by unitarity; it also exhibits, in the ’t Hooft’s limit $N \rightarrow \infty$ with $g^2 N = 2\hat{g}^2$ fixed, confinement with a finite string tension $\sigma = \frac{\hat{g}^2}{2}$. This explains the success of ’t Hooft’s approach in computing the spectrum of the $q\bar{q}$ bound states. The deep reason of this good behaviour lies in the absence of ghosts in this formulation; however there is no smooth way of deriving it from any acceptable gauge choice in higher dimension ($D > 2$). Moreover the confinement exhibited at this stage is, in a sense, trivial, being shared by QED_2 .

We end up with two basically different results for the *same* model and with the *same* gauge choice (LCG), according to the different ways in which IR singularities are regularized. Moreover we are confronted with the following paradox: the prescription which is mandatory in dimensions $D > 2$ is the one which fails at $D = 2$. What is the meaning (if any) of eq.(12)?

III. THE GEOMETRICAL APPROACH

In order to understand this point, it is worthwhile to study the problem on a compact two-dimensional manifold; possible IR singularities will be automatically regularized in a gauge invariant way. For simplicity, we choose the sphere S^2 . We also consider the slightly simpler case of the group $U(N)$. On S^2 we envisage a smooth non self-intersecting closed contour and a loop winding around it a number n of times. We call A the total area of the sphere, which eventually will be sent to ∞ , whereas \mathcal{A} will be the area “inside” the loop we keep finite in this limit.

Our starting point is the well-known heat-kernel expressions [12] of a non self-intersecting Wilson loop for a pure $U(N)$ YMT on a sphere with area A

$$\begin{aligned} \mathcal{W}_n(A, \mathcal{A}) &= \frac{1}{\mathcal{Z}(A)^N} \sum_{R,S} d_R d_S \exp \left[-\frac{g^2 \mathcal{A}}{2} C_2(R) - \frac{g^2 (A - \mathcal{A})}{2} C_2(S) \right] \\ &\times \int dU \text{Tr}[U^n] \chi_R(U) \chi_S^\dagger(U), \end{aligned} \quad (16)$$

$d_{R(S)}$ being the dimension of the irreducible representation $R(S)$ of $U(N)$; $C_2(R)$ ($C_2(S)$) is the quadratic Casimir expression, the integral in (16) is over the $U(N)$ group manifold while $\chi_{R(S)}$ is the character of the group element U in the $R(S)$ representation. $\mathcal{Z}(A)$ is the partition function of the theory, its explicit form being easily obtained from $\mathcal{W}_0(A, \mathcal{A}) = 1$.

We write eq.(16) explicitly for $N > 1$ and $n > 0$ in the form

$$\begin{aligned} \mathcal{W}_n(A, \mathcal{A}) &= \frac{1}{\mathcal{Z}(A)} \sum_{m_i=-\infty}^{+\infty} \Delta(m_1, \dots, m_N) \Delta(m_1 + n, m_2, \dots, m_N) \\ &\times \exp \left[-\frac{g^2 A}{4} \sum_{i=1}^N (m_i)^2 \right] \exp \left[-\frac{g^2 n}{4} (A - \mathcal{A})(n + 2m_1) \right]. \end{aligned} \quad (17)$$

We have described the generic irreducible representation by means of the set of integers $m_i = (m_1, \dots, m_N)$, related to the Young tableaux, in terms of which we get

$$C_2(R) = \frac{N}{24}(N^2 - 1) + \frac{1}{2} \sum_{i=1}^N \left(m_i - \frac{N-1}{2} \right)^2, \quad d_R = \Delta(m_1, \dots, m_N). \quad (18)$$

Δ is the Vandermonde determinant and the integration in eq.(16) has been performed explicitly, using the well-known formula for the characters in terms of the set m_i and taking symmetry into account.

From eq.(17) it is possible to derive, for $n = 1$ and in the large- A decompactification limit, precisely the expression (15) we obtained by resumming the perturbative series in the 't Hooft's approach. This is a remarkable result as it has now been derived in a purely geometrical way without even fixing a gauge. Actually, in the decompactification limit $A \rightarrow \infty$ at fixed \mathcal{A} , from eq.(17) one gets the following expression for any value of n and N [14]

$$\begin{aligned} \mathcal{W}_n(\mathcal{A}; N) &= \frac{1}{nN} \exp\left[-\frac{g^2 \mathcal{A}}{4} n(N+n-1)\right] \\ &\times \sum_{k=0} \frac{(-1)^k \Gamma(N+n-k)}{k! \Gamma(N-k) \Gamma(n-k)} \exp\left[\frac{g^2 \mathcal{A}}{2} n k\right]. \end{aligned} \quad (19)$$

We notice that when $n > 1$ the simple abelian-like exponentiation is lost. In other words the theory starts feeling its non-abelian nature as the appearance of different “string tensions” makes clear. The winding number n probes its colour content. The related light-front vacuum, although simpler than the one in the equal-time quantization, cannot be considered trivial any longer.

Eq.(19) exhibits an interesting symmetry under the exchange of N and n . More precisely, we have that

$$\mathcal{W}_n(\mathcal{A}; N) = \mathcal{W}_N(\tilde{\mathcal{A}}; n), \quad \tilde{\mathcal{A}} = \frac{n}{N} \mathcal{A}, \quad (20)$$

a relation that is far from being trivial, involving an unexpected interplay between the geometrical and the algebraic structure of the theory [14].

Looking at eq.(20), the abelian-like exponentiation for $U(N)$ when $n = 1$ appears to be related to the $U(1)$ loop with N windings, the “genuine” triviality of Maxwell theory providing the expected behaviour for the string tension. Moreover we notice the intriguing feature that the large- N limit (with n fixed) is equivalent to the limit in which an infinite number of windings is considered with vanishing rescaled loop area. Alternatively, this rescaling could be thought to affect the coupling constant $g^2 \rightarrow \frac{n}{N} g^2$.

From eq.(19), in the limit $N \rightarrow \infty$, one can recover the Kazakov-Kostov result [15]

$$\mathcal{W}_n(\mathcal{A}; \infty) = \frac{1}{n} L_{n-1}^{(1)}\left(\frac{\hat{g}^2 \mathcal{A} n}{2}\right) \exp\left[-\frac{\hat{g}^2 \mathcal{A} n}{4}\right]. \quad (21)$$

Now, using eq.(20) we are able to perform another limit, namely $n \rightarrow \infty$ with fixed $n^2 \mathcal{A}$

$$\lim_{n \rightarrow \infty} \mathcal{W}_n(\mathcal{A}; N) = \frac{1}{N} L_{N-1}^{(1)}\left(g^2 \mathcal{A} n^2 / 2\right) \exp\left[-\frac{g^2 \mathcal{A} n^2}{4}\right]. \quad (22)$$

We remark that this large- n result reproduces the resummation of the perturbative series (for any n) (eq.(13)) in the *causal* formulation of the theory.

We go back to the exact expression we have found on the sphere for the Wilson loop (eq.17). As first noted by Witten [16], it is possible to represent $\mathcal{W}_n(A, \mathcal{A})$ (and consequently $\mathcal{Z}(A)$) as a sum over instable instantons, where each instanton contribution is associated to a finite, but not trivial, perturbative expansion. The easiest way to see it, is to perform a Poisson resummation

$$\sum_{m_i=-\infty}^{+\infty} F(m_1, \dots, m_N) = \sum_{f_i=-\infty}^{+\infty} \tilde{F}(f_1, \dots, f_N), \quad (23)$$

$$\tilde{F}(f_1, \dots, f_N) = \int_{-\infty}^{+\infty} dz_1 \dots dz_N F(z_1, \dots, z_N) \exp [2\pi i (z_1 f_1 + \dots + z_N f_N)]$$

in eq.(17). One gets

$$\mathcal{W}_n(A, \mathcal{A}) = \frac{1}{\mathcal{Z}(A)} \exp \left[\frac{g^2 n^2 (A - 2\mathcal{A})^2}{16A} \right] \times$$

$$\sum_{f_i=-\infty}^{+\infty} \exp [-S_{inst}(f_i)] W(f_1, \dots, f_N) \exp \left[-2\pi i n f_1 \frac{A - \mathcal{A}}{A} \right], \quad (24)$$

where

$$S_{inst}(f_i) = \frac{4\pi^2}{g^2 A} \sum_{i=1}^N f_i^2, \quad (25)$$

and

$$W(f_1, \dots, f_N) = \int_{-\infty}^{+\infty} dz_1 \dots dz_N \exp \left[-\frac{1}{g^2 A} \sum_{i=1}^N z_i^2 \right] \exp \left(\frac{i n z_1}{2} \right) \times$$

$$\Delta(z_1 - 2\pi \tilde{f}_1, \dots, z_N - 2\pi \tilde{f}_N) \Delta(z_1 + 2\pi \tilde{f}_1, \dots, z_N + 2\pi \tilde{f}_N), \quad (26)$$

with

$$\tilde{f}_1 = f_1 + \frac{i g^2 n}{8\pi} (A - 2\mathcal{A}).$$

These formulae have a nice interpretation in terms of instantons. Indeed, on S^2 , there are non trivial solutions of the Yang-Mills equation, labelled by the set of integers $f_i = (f_1, \dots, f_N)$

$$\mathcal{A}_\mu(x) = \text{Diag} (f_1 \mathcal{A}_\mu^0(x), f_2 \mathcal{A}_\mu^0(x), \dots, f_N \mathcal{A}_\mu^0(x)) \quad (27)$$

where $\mathcal{A}_\mu^0(x) = \mathcal{A}_\mu^0(\theta, \phi)$ is the Dirac monopole potential,

$$\mathcal{A}_\theta^0(\theta, \phi) = 0, \quad \mathcal{A}_\phi^0(\theta, \phi) = \frac{1 - \cos \theta}{2},$$

θ and ϕ being spherical coordinates on S^2 . The term $\exp[-2\pi i n f_1 \frac{A-\mathcal{A}}{A}]$ in eq.(24) corresponds to the classical contribution of such field configurations to the Wilson loop.

Only the zero instanton contribution should be obtainable by means of a genuine perturbative calculation. Therefore in the following we single out the zero-instanton contribution ($f_q = 0, \forall q$) to the Wilson loop in eq.(24), obviously normalized to the zero instanton partition function [17].

The equation, after a suitable rescaling, becomes

$$\mathcal{W}_n^{(0)}(A, \mathcal{A}) = \frac{1}{\mathcal{Z}^{(0)}(A)} \exp \left[\frac{g^2 n^2 (A - 2\mathcal{A})^2}{16A} \right] W_1(0, \dots, 0) \quad (28)$$

with

$$\begin{aligned} W_1(0, \dots, 0) &= \int_{-\infty}^{+\infty} dz_1 \dots dz_N \exp \left[-\frac{1}{2} \sum_{i=1}^N z_i^2 \right] \exp \left(\frac{in\sqrt{g^2 A} z_1}{2\sqrt{2}} \right) \\ &\times \Delta(z_1 - \frac{in}{4} \sqrt{\frac{2g^2}{A}}(A - 2\mathcal{A}), \dots, z_N) \Delta(z_1 + \frac{in}{4} \sqrt{\frac{2g^2}{A}}(A - 2\mathcal{A}), \dots, z_N). \end{aligned} \quad (29)$$

The two Vandermonde determinants can be expressed in terms of Hermite polynomials [13] and then expanded in the usual way. The integrations over z_2, \dots, z_N can be performed, taking the orthogonality of the polynomials into account; we get

$$\begin{aligned} \mathcal{W}_n^{(0)}(A, \mathcal{A}) &= \exp \left[\frac{g^2 n^2 (A - 2\mathcal{A})^2}{16A} \right] \prod_{n=0}^N \frac{1}{n!} \prod_{k=2}^N (j_k - 1)! \frac{\varepsilon^{j_1 \dots j_N} \varepsilon_{j_1 \dots j_N}}{\mathcal{Z}^{(0)}(A)} \\ &\int_{-\infty}^{+\infty} dz_1 \exp \left[-\frac{1}{2} z_1^2 \right] \exp \left(\frac{in\sqrt{g^2 A} z_1}{2\sqrt{2}} \right) H e_{j_1-1}(z_{1+}) H e_{j_1-1}(z_{1-}), \end{aligned} \quad (30)$$

where

$$z_{1\pm} = z_1 \pm \frac{in}{4} \sqrt{\frac{2g^2}{A}}(A - 2\mathcal{A}). \quad (31)$$

Integration over z_1 finally gives

$$\mathcal{W}_n^{(0)}(A, \mathcal{A}) = \frac{1}{N} L_{N-1}^{(1)} \left(\frac{g^2 \mathcal{A} (A - \mathcal{A}) n^2}{2A} \right) \exp \left[-\frac{g^2 \mathcal{A} (A - \mathcal{A}) n^2}{4A} \right]. \quad (32)$$

At this point we remark that, in the decompactification limit $A \rightarrow \infty$, \mathcal{A} fixed, the quantity in the equation above *exactly* coincides, for any value of N , with eq.(13), which was derived following completely different considerations. We recall indeed that eq.(13) was obtained by a full resummation of the perturbative expansion of the Wilson loop in terms of causal Yang-Mills propagators in LCG. Its meaning is elucidated by noting that it just represents the zero-instanton contribution to the Wilson loop, a genuinely perturbative quantity [17].

In turn it also coincides with the expression of the exact result in the large- n limit, keeping fixed the value of $n^2\mathcal{A}$ (eq.(22)). This feature can be understood if we remember that instantons have a finite size; therefore small loops are essentially blind to them [14].

If the perturbative result has been correctly interpreted, it should be related to the local behaviour of the theory: in particular it should be possible to derive it starting from any topology, when the decompactification limit is eventually performed. In [18] the zero-instanton contribution to a homologically trivial Wilson loop on the torus T^2 has been computed: the Poisson resummation is harder there and a larger number of classical solutions complicates the geometrical structure. In spite of the different complexity, the zero-instanton contribution, in the decompactification limit, still coincides with the perturbative result, as expected.

IV. THE k -SECTORS

It was first noticed by Witten [19] that two-dimensional Yang-Mills theory and two-dimensional QCD with adjoint matter do possess k -sectors. We consider $SU(N)$ as the gauge group: since Yang-Mills fields transform in the adjoint representation, the true local symmetry is the quotient of $SU(N)$ by its center, Z_N . A standard result in homotopy theory tells us that the quotient is not simply connected, the first homotopy group being $\Pi_1(SU(N)/Z_N) = Z_N$. This result is of particular relevance for the vacuum structure of a two-dimensional gauge theory: in the case at hand we have exactly N inequivalent quantiza-

tions, parametrized by a single integer k , taking the values $k = 0, 1, \dots, N-1$. Concerning the pure $SU(N)$ Yang-Mills theory, the explicit solution when k -states are taken into account was presented in Ref. [20]: their main result, the heat-kernel propagator on the cylinder, allows to compute partition functions and Wilson loops on any two-dimensional compact surface, therefore generalizing the well-known Migdal's solution [12] to k -sectors. Wilson loops, in this case, strongly depend on k : for a non self-intersecting loop we have, on the plane,

$$\mathcal{W}_k(A) = \frac{1}{N^2 - 1} \left[1 + \frac{kN(N+2)(N-k)}{(k+1)(N-k+1)} e^{-\frac{q^2 A}{2}(N+1)} \right. \quad (33)$$

$$\left. + \frac{(N+1)(N-k-1)}{k+1} e^{-\frac{q^2 A}{2}(N-k)} + \frac{(N+1)(k-1)}{N-k+1} e^{-\frac{q^2 A}{2}k} \right]. \quad (34)$$

This result can be obtained starting from the true $SU(N)/Z_N$ theory on the sphere [21], in the decompactification limit, or directly on the plane, using the procedure of [22], working with $SU(N)$ and simulating the k -sectors with a Wilson loop at infinity in the k -fundamental representation. The very same result can be obtained through a perturbative resummation [21] with 't Hooft potential and the k -loop at infinity. On the other hand we expect that the truly perturbative physics ignore the existence of the k parameter: by Poisson-resumming the result on the sphere for $SU(N)/Z_N$, we arrive to an instanton representation different from the $SU(N)$ case. Contribution from N inequivalent classes of instantons ensues, with instanton numbers generalized to rational values by the effect of k . The zero-instanton limit does not depend on k and still reproduces the WML computation (in the decompactification limit) *without* the loop at infinity [21]

$$\mathcal{W}_k^{(0)}(A) = \frac{1}{N+1} + \frac{N}{\mathcal{Z}(N+1)} \int_{-\infty}^{+\infty} dz_1 \dots dz_N \exp \left[-\frac{1}{2} \sum_{j=1}^N z_j^2 \right] \times \quad (35)$$

$$\exp \left[ig(z_1 - z_2) \sqrt{\frac{A}{2}} \right] \Delta^2(z_1, \dots, z_N),$$

where $\mathcal{Z} = \int \mathcal{D}F \exp(-\frac{1}{2} \text{Tr} F^2)$. In the presence of a k -loop at infinity, the WML computation, although coinciding with the zero-instanton limit of the quantum average of two nested (k and adjoint) loops, does depend on k . We conclude that *only* for the complete theory on

the plane (i.e. full-instanton resummed and then decompactified) the equivalence between k -sectors and theories with k -fundamental Wilson loops at infinity holds.

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